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David M. Gómez, Abelino Jiménez, Roberto Bobadilla, Cristián Reyes, Pablo Dartnell

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David M. Gómez[1], Abelino Jiménez[1], Roberto Bobadilla[2], Cristián Reyes[3], Pablo Dartnell[123]

(1) Centro de Investigación Avanzada en Educación (CIAE), Universidad de Chile.

(2) Departamento de Ingeniería Matemática (DIM), Universidad de Chile.

(3) Centro de Modelamiento Matemático (CMM), Universidad de Chile.

Abstract

The application to rational numbers of the procedures and intuitions proper of natural numbers is known as Natural Number Bias. Research on the cognitive foundations of this bias suggests that it stems not from a lack of understanding of rational numbers, but from the way the human mind represents them. In this work, we presented a fraction comparison questionnaire to 502 school children from 5th to 7th grade to investigate if the Natural Number Bias succeeds in explaining their error patterns. About 25% of children responded in a way perfectly consistent with the Bias, but good students committed many errors in items that the Bias predicts to be easy. We propose an explanation based on comparison strategies and wrong generalizations of a common remark used for teaching fraction magnitude.

Introduction

Learning fractions is an important challenge within the middle school curriculum. Fractions are typically the very first approach of school children to number systems beyond that of natural numbers. To master fractions, students must learn new concepts, procedures, and intuitions that often contradict their accumulated knowledge of the natural number system. For instance, the multiplication of two fractions may be smaller than the intervening factors (e.g. $1/2 \times 1/4 = 1/8$), and any fraction can be written in infinitely many equivalent ways (e.g. $1/2 = 2/4 = 3/6 = \ldots$). Many students fail to understand fractions even at the most basic levels, something problematic under the light of recent evidence linking successful learning of fractions to advanced topics like algebra (Booth & Newton, 2012). Although the lack of appropriate mathematical knowledge by many teachers is a very important factor contributing to this failure (e.g. Valdemoros Alvarez, 2010), other less evident factors may also play a relevant role.

Ni and Zhou (2005) presented a review about a particular type of frequent errors linked to the understanding of fractions. These errors seemed to stem from the generalization to fractional contexts of the concepts, procedures, and intuitions proper of natural numbers. The authors used for them the umbrella term Natural Number Bias. Typical examples associated to this bias are reasoning that $2/3 < 2/5$ because $3 < 5$, as well as computing $1/2 + 1/3 = 2/5$, thinking that processing separately both fraction components (numerators and denominators) is enough for obtaining the desired result. Errors due to reasoning on the basis of natural number knowledge are not limited to calculation procedures: school children from 7^{th} to 11^{th} grade may state that there is a finite number of rationals between 1/5 and 4/5, as if rational numbers possessed a successor (Vamvakoussi & Vosniadou, 2010). It is important to underline, as Ni and Zhou do, that these errors are not simply the result of a failed learning experience but reflections of the deep, intuitive way in that the human mind deals with fractions even in adults who work proficiently with rational numbers. Recent investigations have demonstrated this by presenting questionnaires in which pairs of fractions such as 2/7 and 5/7, or 3/5 and 3/8, must be compared. These two fractions pairs may be called congruent and incongruent respectively because of the relation between the magnitude of the fractions and the magnitude of the natural numbers composing them. In this sense, in the former pair the greatest fraction has the greatest numerator so that fractional and natural magnitudes point in the same direction, whereas in the latter pair the greatest fraction has the least denominator and hence fractional and natural magnitudes point in opposite directions. When using these types of items, adults (Vamvakoussi, Van Dooren, & Verschaffel, 2012) and even expert mathematicians (Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013) respond more slowly to incongruent fraction pairs.

The present work explores the extent to which the Natural Number Bias provides a useful account of the errors committed by a sample of 5^{th}- to 7^{th}-grade children in a fraction comparison questionnaire. To do this, we selected fraction pairs that allowed us to contrast explicitly the congruent/incongruent dimension. In addition, based on research on the neural processing of fractions (e.g. Barraza, Gómez, Oyarzún, & Dartnell, under review; Ischebeck, Schocke, & Delazer, 2009), we selected fraction pairs that either have or have no common components. Pairs with common components (e.g. 2/7 and 5/7) tend to be compared by just looking at the non-common component (see also Bonato, Fabbri, Umiltà,

& Zorzi, 2007), whereas pairs lacking common components (e.g. 2/3 and 1/4) require different strategies such as computing cross multiplications or estimating the numerical magnitude of each fraction.

Methods

Participants

Five hundred and two school children of 5th (n = 165), 6th (n = 181), and 7th (n = 156) grade classes from five schools located in different areas of Santiago, Chile, participated in this study. All children were authorized by their parents' signature of an informed consent form.

Questionnaire

We selected 24 fraction pairs grouped according to two factors: the presence or lack of common components, and congruency/incongruency (see Table 1). We classified a fraction pair a/b and c/d as congruent if $a \leq c$, $b \leq d$, and $a/b \leq c/d$ (or vice versa); or as incongruent if $a \leq c$, $b \leq d$, and $a/b \geq c/d$ (or vice versa). In other words, congruent pairs are those in which the greatest numerator and the greatest denominator both belong to the greatest fraction, whereas in incongruent pairs the greatest numerator and the greatest denominator both belong to the least fraction.

	With common components		Without common components	
Congruent pairs	4/9, 8/9	9/11, 4/11	5/7, 1/3	5/16, 12/17
	7/19, 15/19	15/17, 6/17	3/14, 9/17	10/17, 3/9
	2/11, 3/11	7/8, 4/8	2/5, 11/18	17/19, 4/9
Incongruent pairs	4/15, 4/6	1/9, ¼	5/6, 8/19	2/4, 3/13
	5/8, 5/17	3/7, 3/14	6/13, 4/5	6/18, 5/6
	7/15, 7/10	6/14, 6/8	2/3, 5/17	4/15, 2/5

Table 1: Full item list of the fraction comparison questionnaire.

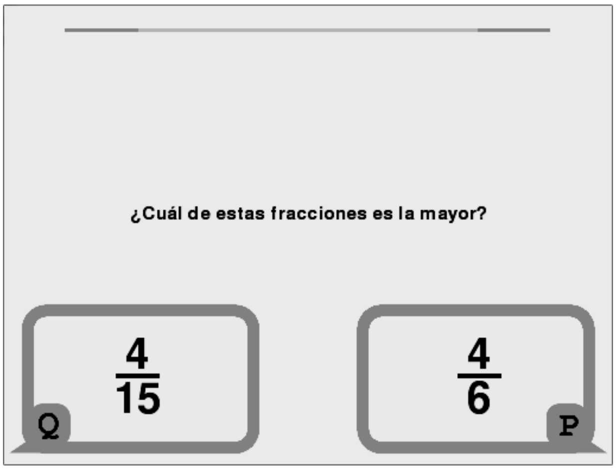

Figure 1: Screen capture of an item of the questionnaire. On top, a colored bar indicates time left for answering. At the bottom, the fraction pair to be compared.

Mathematics achievement

We measured children's general mathematics knowledge by means of tests that their schools apply every year. As the five selected schools share a common curriculum and instructional design, these tests were the same for all schools but differed for each grade. Because of this, we normalized children's scores on a grade-by-grade basis by subtracting the average score and dividing for their standard deviation. We were only able to obtain these test scores for 451 children out of the total sample.

Procedure

Each class was tested in the Computer Science classroom of their school. The questionnaire was presented by computer and programmed in Python+PyGame. Each child worked individually. All items presented the question "Which of these fractions is the greatest?" ("¿Cuál de estas fracciones es la mayor?") at the middle of the screen, whereas the fractions to be compared were displayed at the bottom (Figure 1). Children pressed the keys Q or P to select the left or right fraction as the greatest, respectively. Items not answered within 10 seconds of presentation were considered as omitted, and the next item was then presented. A color-changing bar on top of the screen displayed the time left for answering.

Children were aware that their outcome in this questionnaire would not have effect on their school grades. We asked them to answer each item carefully, and to follow their intuition in case of doubt.

Results

Overall mean accuracy, computed as the ratio of correct responses to non-omitted items, was 59.0% (SD = 17.6). Differences among the three grades were negligible (5th grade: 58.7%, 6th grade: 60.0%, 7th grade: 58.1%; $F(2,499) = 0.51$, $p = .60$). Accuracy scores correlated significantly with general mathematics knowledge, with a weaker effect in 7th

grade than in 5th and 6th grades (5th grade: $r=.37$, $t(127)=4.5$, $p<.001$; 6th grade: $r=.40$, $t(175)=5.7$, $p<.001$; 7th grade: $r=.26$, $t(143)=3.2$, $p=.002$). Cronbach's α was .76, suggesting a good (though not excellent) degree of internal consistency.

	With common components	Without common components	Mean
Congruent	82.2%	72.4%	77.3%
Incongruent	41.2%	40.1%	40.6%
Mean	61.7%	56.3%	59.0%

Table 2: Average scores in the fraction comparison questionnaire.

Table 2 presents accuracy rates per item types. A 2-way ANOVA showed a statistically significant effect of the presence or absence of common components: fraction pairs with common components were answered in average 5% better than pairs without common components ($F(1,1503)=11.1$, $p<.001$). Congruency has a much larger effect, with congruent items being answered in average 36.7% better than incongruent items ($F(1,1503)=505.3$, $p<.001$). There was a statistically significant interaction between these factors as well, indicating that the difference in accuracy of congruent over incongruent items was larger in fraction pairs with common components (difference for items with common components: 41.0%; without: 32.3%; $F(1,1503)=7.0$, $p=.008$).

An item-per-item analysis of accuracy rates shows that this pattern of results is consistent across all 24 items of the questionnaire (Figure 2A), in close agreement with the predictions of the Natural Number Bias.

Children who were 100% accurate in congruent items and 0% in incongruent items represent extreme cases of the Bias. We observed 126 children (25.1% of the sample) that answered the questionnaire in this way: 46 in 5th grade (27.9%), 40 in 6th grade (22.1%), and 40 in 7th grade (25.6%).

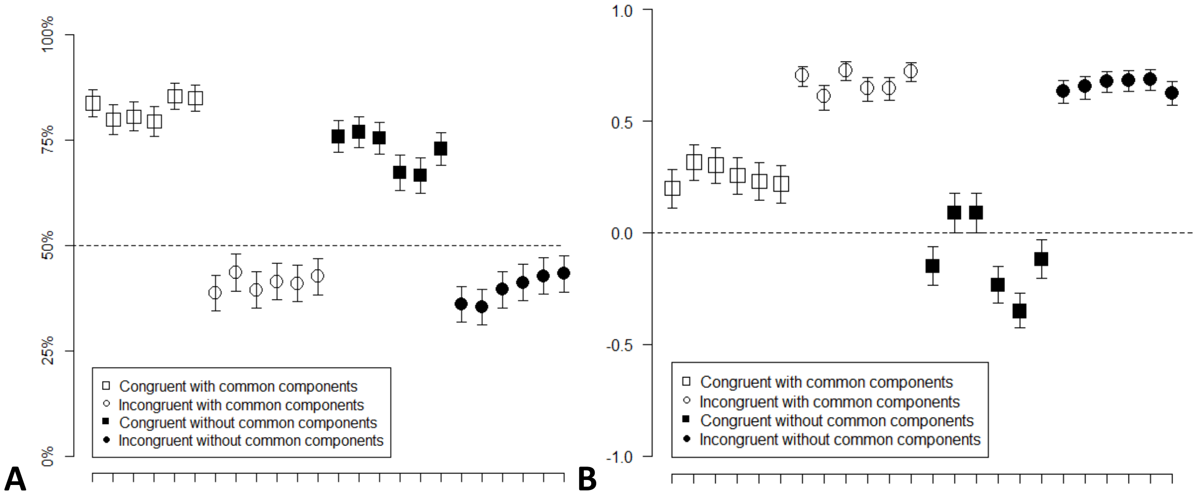

Figure 2: (A) Average scores per item. (B) Item-total correlations. Vertical bars depict 95% confidence intervals.

To further explore our different item types, we computed item-total correlations. Figure 2B depicts correlations of all 24 individual items and the total questionnaire scores. High correlations indicate that children who answered those items correctly tend to have high overall scores, and vice versa. That is, items with high item-total correlation are considered as measuring in an appropriate way overall knowledge of fraction comparison. Figure 2B shows several interesting features. First, all incongruent items present the highest correlations (ranging from .61 to .73) regardless of the presence of common components. This is in line with the predictions of the Natural Number Bias, which implies that incongruent items are the hardest for children. Second, congruent items with common components, that is to say fraction pairs that share a common denominator, are also positively correlated but to a smaller degree (ranging from .20 to .32). This may be due to the fact that these items are answered correctly by the vast majority of children (the average score being 82.2%), thus being unable to discriminate between children with good and bad overall scores. The final and most intriguing feature of Figure 2B is that congruent items without common components display very low or even negative correlations (ranging from -.35 to .09), signalling that these items do not align well with the rest of the questionnaire. To some extent, this reflects the large share of extremely biased children who get scores lower than average (they have overall scores about 50%) but all congruent items correct. Removing these children from the sample, however, does not alter the overall pattern (Table 3). This suggests that students with better general mathematics knowledge might be doing worse than average in these items, which indeed turns out to be the case: Students who were 1.5 standard deviations or more above average in an independent test (this amounts to 27 children out of the 451 for whom it was possible to obtain test scores) have an average score in comparing congruent items without common components of 58.0%, lower than the general average of 72.1%. Thus, top students behaved in a way opposite to the predictions of the Natural Number Bias for the case of items with no common components (their average scores in all other item types are above 80%).

Item	Correlation (full sample)	Correlation (subsample)	Are the two greatest naturals part of the same fraction?
5/7, 1/3	-.15	-.05	Yes
5/16, 12/17	.09	.21	No
3/14, 9/17	.09	.21	No
10/17, 3/9	-.24	-.13	Yes
2/5, 11/18	-.35	-.27	Yes
17/19, 4/9	-.12	-.02	Yes

Table 3: Low and negative item-total correlations for congruent items without common components are observed both in the full sample and after remotion of the 126 extremely biased children. The rightmost column classifies items according to the location of the two largest natural numbers in the item.

Discussion

We presented a fraction comparison questionnaire to 5th-, 6th-, and 7th-grade children in order to explore their pattern of responses and contrast it with the predictions of the Natural Number Bias. To do this, we included fraction pairs classified as congruent or incongruent according to the relation between their correct answers and the answers that would be obtained by focusing only on the natural numbers composing them. The Natural Number Bias predicts that congruent items get higher scores systematically. Our results present both support and challenges for the Natural Number Bias account. On the full sample average, congruent items had scores substantially higher than those of incongruent items (average difference of 36.7%). Moreover, a group of about 25% of the sample and approximately equally distributed among the different grades, answered the questionnaire in total agreement with the Bias. Although our data do not allow us to distinguish whether these extreme cases are due to failures in learning fraction comparison or in retaining this knowledge, they do suggest that a sizable number of children will rely on their natural number knowledge and intuitions when facing uncertainty.

Beyond the good value of Cronbach's α for our questionnaire, item-total correlations contribute importantly towards a clearer picture of children's thought processes. As expected according to the Natural Number Bias, incongruent items are highly predictive of total scores. Congruent items in general present lower correlations, even close to zero or negative in the case of items with no common components. These negative correlations are not simply due to that 25% of the sample who responded in complete agreement with the Bias, since the pattern of correlations still appears when looking at the other 75%. In an intriguing finding, we discovered that the top 6% students obtain lower than average scores in these items. Careful observation of the fraction pairs in these items shows that pairs with negative item-total correlations share a common feature: The greatest fraction of each pair not only contains the greatest numerator and the greatest denominator, but also these two numbers were the two greatest among all naturals present in the item (Table 3). Such fraction pairs may be called “strongly congruent”, as they are a subset of congruent pairs. Our questionnaire was not designed to study them in detail, a gap that future research should explore.

How are top students thinking, that they end up performing worse than average specifically in congruent items with no common components? A first observation is that they are not applying a single method such as cross multiplication to all items without common components, because they fail in congruent items but answer correctly incongruent ones. Alternatively, they might be aware that natural numbers may be misleading in a fractional context and answering incorrectly because of an excess of caution. This account, however, also fails to explain why they do well in incongruent items without common components. Another possibility that does explain this difference is that top students might be applying a heuristic method, namely that the greatest fraction tends to be the one with the least denominator. Given our selection of fraction pairs without common components, such heuristic leads exactly to good results in incongruent items and to bad results in congruent ones. This can be seen as an overgeneralization of the common remark made by teachers that the magnitude of a fraction grows if its denominator shrinks, and vice versa. This reasoning, which is perfect when referring to a single fraction, becomes a heuristic when applied to a fraction comparison item because in this new context it may systematically lead to wrong answers.

Understanding the patterns of reasoning behind children’s answers in a test is a powerful aid for the design of pedagogical interventions. Understanding common mistakes also allows providing appropriate corrective feedback. Our work thus highlights the importance of taking into account the Natural Number Bias and its strong influence in 5th-, 6th-, and 7th-grade children. Our quantitative approach, and the large sample size considered, did not allow us to focus on subtle factors such as the variety of strategies that each child may use to solve each item (e.g. Clarke & Roche, 2009). In this sense, qualitative data would make a great complement to the data here presented and may shed light on the thinking processes of top students that lead them to perform worse than average in a specific item type.

To what extent it is possible to overcome the Natural Number Bias by means of pedagogical interventions is an open question, although recent research suggests that it is not possible to do it perfectly: Remnants of biased thinking remain in adulthood (Vamvakoussi et al., 2012) and even in expert mathematicians (Obersteiner et al., 2013). Furthermore, other researchers suggest that this Bias partly stems from the way of writing fractions and its use of natural numbers (such as “2” and “3” in “2/3”; see Kallai & Tzelgov, 2012; Mena-Carrasco, Gómez, Araya, & Dartnell, under review), a conclusion that, if true, states that the effects of the Natural Number Bias in fractional tasks is unavoidable. Pedagogical intervention could still, in this case, aim at making students aware of the faulty reasoning behind the Bias.

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References

Barraza, P., Gómez, D. M., Oyarzún, F., & Dartnell, P. (under review). Long-distance neural synchrony correlates with cognitive strategies to compare fractions.

Bonato, M., Fabbri, S., Umiltà, C., & Zorzi, M. (2007). The mental representation of numerical fractions: Real or integer? Journal of Experimental Psychology: Human Perception and Performance, 33(6), 1410-1419.

Booth, J. L. & Newton, K. J. (2012). Fractions: Could they really be the gatekeeper's doorman? Contemporary Educational Psychology, 37(4), 247-253.

Clarke, D. M. & Roche, A. (2009).Students' fraction comparison strategies as a window into robust understanding and possible pointers for instruction. Educational Studies in Mathematics, 72(1), 127-138.

Ischebeck, A., Schocke, M., & Delazer, M. (2009). The processing and representation of fractions within the brain: An fMRI investigation. NeuroImage, 47(1), 403-413.

Kallai, A. Y. & Tzelgov, J. (2012).When meaningful components interrupt the processing of the whole: The case of fractions. Acta Psychologica, 139(2), 358-369.

Mena-Carrasco, G., Gómez, D. M., Araya, R., & Dartnell, P. (under review). Congruity effects in the Numerical Stroop task and fraction comparison competence.

Ni, Y. & Zhou, Y.-D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of Whole Number Bias. Educational Psychologist, 40, 27-52.

Obersteiner, A., Van Dooren, W., Van Hoof, J., & Verschaffel, L. (2013). The natural number bias and magnitude representation in fraction comparison by expert mathematicians. Learning and Instruction, 28, 64-72.

Valdemoros Alvarez, M. E. (2010). Dificultades experimentadas por el maestro de primaria en la enseñanza de fracciones. Relime, 13(4-II), 423-440.

Vamvakoussi, X., Van Dooren, W., & Verschaffel, L. (2012). Naturally biased? In search for reaction time evidence for a natural number bias in adults. Journal of Mathematical Behavior, 31, 344-355.

Vamvakoussi, X. & Vosniadou, S. (2010). How many decimals are there between two fractions? Aspects of secondary school students' understanding of rational numbers and their notation. Cognition and Instruction, 28(2), 181-209.