13th International Congress on Mathematical Education
Hamburg, 24-31 July 2016

CLUSTERING ANALYSIS AS A WINDOW INTO CHILDREN'S STRATEGIES FOR COMPARING FRACTIONS

David M. Gómez, Pablo Dartnell

Universidad de Chile

Individual differences are ubiquitous in human cognition, and learning Mathematics is not an exception. In this work, we studied the biases and strategies deployed by middle school children (N=490) in a computerized fraction comparison task. Whereas overall results suggest the presence of a strong bias for mistaking component magnitude for fraction magnitude, a clustering analysis reveals the coexistence of at least five distinct manners of reasoning. This implies that cognitive models of the learning of fractions must take into account individual variability in terms of strategy use. Furthermore, discovering such groups holds promise not only for research purposes, but also as a contribution to teaching practices because of its potential to expose common mistake patterns.

INTRODUCTION

An important constituent of mathematical competence is the capacity of reasoning flexibly using a set of basic concepts and procedures. Even basic arithmetic operations are subject to be performed with a variety of strategies (e.g. Ashcraft, 1990), which may be used to different extents by different learners. Children may use different strategies when solving repeatedly problems of the same type, even when one strategy is systematically more convenient than the others (e.g. Siegler & Stern, 1998). Sometimes, deep misconceptions may be revealed by the incorrect use of some strategies, or by the wrong application of some other ones. Handling these misconceptions requires important efforts from educators, who must be able to identify and then correct them. The degree of variability in learners' conceptions and strategies—and the relevance of detecting misconceptions—grows importantly when learners have multiple representations available, as it is the case of learning fractions and rational numbers. Rationals may be seen as ratios, operators, quotients, or measures, among other possibilities (Kieren, 1976).

A necessary step in learning fractions successfully is the ability to distinguish between the magnitudes of fractions themselves and that of the natural numbers that compose them. Many of the most common mistakes that children commit when reasoning about rational numbers involve the overgeneralization of facts, strategies, and procedures valid for natural numbers. For instance, when 55% of a sample of 13-year-olds and 36% of a sample of 17-year-olds claim that 7/8 + 12/13 is approximately 19 or 21 (Reys, Rybolt, Bestgen, & Wyatt, 1982). This tendency to overgeneralization has been termed *whole number bias* or *natural number bias* (e.g. Ni & Zhou, 2005; Van Dooren, Lehtinen, & Verschaffel, 2015). Research with children demonstrates that the distinction between components' magnitudes and fraction magnitude is hard to achieve (Gómez, Jiménez, Bobadilla, Reyes, & Dartnell, 2014; Stafylidou & Vosniadou, 2004), and even data from educated adults (DeWolf & Vosniadou, 2015; Vamvakoussi, Van Dooren, & Verschaffel, 2012) as well as expert mathematicians (Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013) provide evidence of interference between both types of magnitude.

To study the conflict between the magnitudes of fractions and those of the fractions' components, many of these studies have used fraction comparison tasks where the concept of *congruency* plays a central role. As depicted in Figure 1, a pair of fractions is called *congruent* if the numerical magnitudes of the fraction, the numerator, and the denominator are maximized by the same fraction, for instance 1/4 and 2/5. *Incongruent* pairs, in contrast, are those in which the numerator and the denominator of the smaller fraction are larger, like 1/3 and 2/9. If one compares fractions by direct comparison of the magnitudes of the components only, then congruent items will be answered correctly and incongruent items incorrectly.

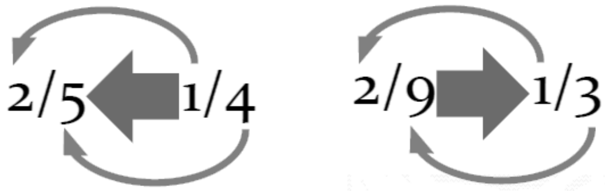

Figure 1: Examples of fraction pairs representing congruent (left) and incongruent (right) comparison items. Thin arrows point to the largest components in each pair, and thick arrows point to the largest fraction. In the congruent case, all arrows point in the same direction, whereas in the incongruent case the thin and the thick arrows point in opposite directions.

Although studies with school children (e.g. Gómez et al., 2014) have shown that indeed congruency may play a relevant role in shaping children's answers in a fraction comparison task, the notion of congruency itself seems not to stem from the mathematics education literature. The concept might have been introduced by a neuroimaging study conducted by Ischebeck, Schocke, and Delazer (2009), grounded in the psychological tradition of studying the cognitive conflict produced when two sources of information clash in a decision-making task such as the classical Stroop task (when presented with the word *red* written in blue ink and asked which ink color was used, literate people may mistakenly respond "red" instead of "blue"). However, evidence that students' accuracy patterns align with congruency is not—by itself—a demonstration that congruency is a cognitively or educationally relevant construct. In other words, a correlation between congruency and students' performance does not imply that congruency causes some patterns of reasoning about fractions. Nonetheless, this alignment suggests that congruency may be closely related to some other, cognitively and educationally relevant, variables yet to be discovered and understood.

A highly relevant aspect to note is that most studies addressing the role of congruency and other stimuli dimensions in fraction comparison have so far focused in overall group performance, that is to say, in the average performance across participants. This decision implicitly assumes that the average performance is representative of that of each participant, so that conclusions drawn from average values reflect the cognitive processes engaged by all participants. However, it is well known that this assumption might be easily violated. Variables might have a bimodal distribution, rendering their overall average values meaningless by themselves. Hence, only a careful examination of individual differences can provide strong support for cognitive and educational theories about children's reasoning about fractions. In line with Clarke and Roche (2009), in this article we focus on the strategies used by children to compare pairs of fractions, although with a quantitative, clustering-based approach. More specifically, we present the results of a clustering analysis conducted on the data collected by Gómez et al. (2014) to look for groups of children according to their strategies for comparing fractions. The outcome of this analysis will then be used to infer individual variability in terms of strategy use.

METHODS

Participants. Five hundred and two children of 5th, 6th, and 7th grade classes participated in this study (approximate age range 10-12 years old). Children belonged to five schools of the Greater Santiago area in Chile. Informed consent was obtained from a parent or tutor of each child prior to testing.

Fraction comparison task. The task consisted of 24 pairs of fractions to be compared, and was presented to children by computer. Items were equally distributed between congruent and incongruent. In addition, within each category half of the pairs shared a common numerator or denominator while in the other half they shared no common component. Each item had a time limit of 10 s to be answered and items not answered within this time were considered omitted, leading to an approximate total duration of 4 minutes. We refer the reader to Gómez et al. (2014) for a more detailed description of the task and the full list of items.

Procedure. Children were tested in groups, in the computer classroom of each school. Each child worked individually. At least two researchers or research assistants monitored each testing session.

Clustering analysis. We discarded twelve children who took longer than the time limit in three or more items. For the other 490 children, we computed percentages of correct answers for the four item types (congruent/incongruent, with/without common components) and used these scores to run a k-means clustering algorithm to group children into k groups. We tried all values of k from 2 to 10 and selected k = 6 as the best solution, based on an index calculated using percentages of explained variance and average standard deviations per cluster, among other measures.

RESULTS

Table 1 presents the results of the clustering analysis. We found two groups whose answers are highly influenced by congruency, A and E. Children in the former group answered, as expected, most congruent items correctly and most incongruent items incorrectly, whereas the latter showed a less extreme pattern but in the opposite direction, with incongruent items having better scores than congruent ones.

The second largest group was B, gathering children with the highest overall scores. This group, however, presents a drop of about 20% in performance when comparing congruent items without common components. A similar pattern of answers, but with a much more pronounced drop for the same item type, is exhibited by group D.

Group C comprised children who had relative success in comparing fractions when these had a common component, but failed when these shared no common component. Finally, group F showed a pattern of answers below the 50% level for all item types.

		With a common component		Without common components	
Group	**Size**	**Congruent**	**Incongruent**	**Congruent**	**Incongruent**
A	245	93 %	4 %	97 %	2 %
B	84	94 %	94 %	74 %	93 %
C	48	84 %	67 %	57 %	55 %
D	43	89 %	94 %	18 %	96 %
E	38	20 %	85 %	15 %	90 %
F	32	38 %	25 %	48 %	29 %

Table 1: Group sizes and scores for all item types.

DISCUSSION

The clustering analysis revealed important differences among children's patterns of answers in the fraction comparison task. Importantly, these differences reflect distinct patterns of reasoning for comparing fractions, constituting a relevant step forward with respect to previous analyses that focused only in averages across all children (e.g. Gómez et al., 2014).

The majority of children (group A, half of the sample) compared fractions guided by congruency, that is to say selecting as the larger fraction the one with larger components. This pattern of reasoning was expected, given previous evidence in children's and adults' data (Gómez et al., 2014; Obersteiner et al., 2013; Vamvakoussi et al., 2012). However, a smaller group (E, 8% of the sample) showed an opposite behavior, selecting as the larger fraction the one with smaller components. Answers from the latter group, although unexpected from the viewpoint of congruency, align with children's self-reports in a previous study (Stafylidou & Vosniadou, 2004).

Groups B and D (comprising 17% and 9% of the sample, respectively) showed similar patterns of scores, answering about 90% correctly for all item types except for congruent items without common components. Gómez et al. (2014) proposed that a tendency to select the fraction with a smaller denominator as larger may lead to this pattern of results. This heuristic is moot with respect to items where fractions share a common denominator, but given our definition of congruency, it answers correctly all incongruent items and incorrectly all congruent items without common components. Assuming that children answer items with a common denominator by resorting to an *ad hoc* strategy, scores of children in group D align closely with the proposed heuristic. Group B, instead, may be described as showing a mixed behavior between answering correctly all item types and a tendency to use the described denominator-size heuristic. An interesting question for further inquiry would be to examine how clearly distinguishable groups B and D are. It may be that both groups switch between the same pair of strategies/heuristics, only differing in the probabilities of activation that they assign to each of them. Alternatively, both groups may show qualitative differences in their conceptual understanding of fractions or fraction comparison.

Scores for children in group C (10% of the sample) suggest that they succeed in comparing only fraction pairs that share a common component, and within these items they show a clear congruency effect. These children seem to have the basic knowledge of what fractions are, allowing them to reason about items with a common component, but to lack knowledge and strategies for the more complex cases. Finally, group F (7% of the sample) displays a pattern of scores difficult to interpret without additional data. It is possible that the clustering algorithm formed this group by pooling together all children who did not fit neatly into the first five groups.

A couple of the outcomes from our analysis call into question the idea that children are biased towards congruency, as it has been suggested based on data from previous studies. First, the presence of group E, where children's answers point in the direction opposite to the one predicted by congruency. Second, the drops in performance exhibited by groups B and D occur specifically for a subset of congruent items, namely those lacking common components. These results suggest, at least, that a bias towards congruency is not the main driver of children's reasoning in many cases. Indeed, the use of specific heuristics or strategies for comparing fractions may provide a better explanation for some of our observations. Children in group A may answer according to the rule "larger components, larger fraction"; those in group E according to "smaller components, larger fraction"; and those in group D according to "smaller denominator, larger fraction". Still, congruency might play a considerable role in this context by explaining why group A outnumbers group E, or why there seems not to be an equivalent of group C with incongruent items answered better than congruent ones. We refer the reader to Gómez and Dartnell (2015) for another argument questioning the cognitive relevance of congruency for fraction comparison. Altogether, these findings indicate the need for a reconsideration of the relevance of congruency from the perspectives of both cognitive processing and mathematics education.

ADVANTAGES AND LIMITATIONS

In this article, we partly described the high variability exhibited by 5^{th} to 7^{th} grade children in regard to the biases and strategies they use to compare fractions. We note that the fraction comparison task, despite its simplicity and brevity, was able to show a rich variety of possible patterns of reasoning. The task consisted of 24 items, each of them with a time limit of 10 seconds to be answered. This implies that the total test duration was about four minutes. As the task was presented to children by computer, it was possible to test many children in parallel, simplifying data collection as well.

The computerized application of the fraction comparison task, as well as the absence of pencil and paper for calculations and the time limitation to answer each item, contributed importantly to bring to light children's basic intuitions about fractions. Indeed, in this context children had to answer spontaneously and without recur to written algorithms. Nonetheless, this advantage comes with a price: using this format of task presentation likely poses a heavier burden on students with a low working memory capacity. These students would probably benefit from, for instance, pencil and paper and being able to write down some calculations. The lack of such support might lead to underestimation of the conceptual understanding of fractions that these children have. A similar argument could be made for children with mathematics anxiety. A review by Ashcraft and Krause (2007) shows that persons with mathematics anxiety display lower outcomes in tasks of middle to high difficulty—as it is the case for fraction comparison—. Thus, a relevant issue for future research is to determine how are the results of fraction comparison tasks affected by presentation format (e.g. contrasting the format used in the present study to a standard pencil and paper test).

IMPLICATIONS

In sum, we observed important variability among middle school students with respect to the biases and strategies that they deploy in a fraction comparison task. We stress the fact that our task was very short: 24 items with a time limit of 10 s for each of them, for a maximum total task duration of about 4 minutes. In addition, since the task was presented by computer, it can be applied to several students in parallel if desired.

The possibility to discover different patterns of reasoning based on such a short task are relevant for research purposes and educational practice. Qualitative research in mathematics education (e.g. that involving guided interviews) is highly demanding in terms of the time and human resources needed for data collection. A method like the one described above may be used as a starting point to build screening tasks to obtain potential groups of children reasoning in distinct manners. Participants for guided interviews may then be selected by sampling all different groups, allowing researchers to reach minority groups with significantly less effort. Also neuroscientific research in mathematics education may benefit from screening participants in search of patterns of reasoning: A statistically significant activation in a given brain area (Ischebeck et al., 2009), or a given event-related potential (e.g. Barraza, Gómez, Oyarzún, & Dartnell, 2014; Zhang et al., 2012), can only be interpreted adequately if all experimental participants are assumed to engage similar cognitive processes in solving a mathematical task, which is unlikely if different groups of participants show distinct patterns of reasoning.

Last but not least, the approach presented in this article may prove useful for educational practice as well. A brief fraction assessment like the one presented here allows the teacher to identify the patterns of reasoning most likely engaged by each child. This knowledge may lead to better identification of erroneous patterns of reasoning and their underlying misconceptions, leading in turn to more focused and effective pedagogical action.

ACKNOWLEDGEMENTS

The authors are grateful to the children, parents, and schools who participated in this study. This research was supported by CONICYT programs PIA/Basal FB0003 and PAI/Academia 79130029.

References

Ashcraft, M. H. (1990). Strategic processing in children's mental arithmetic: A review and proposal. In D. F. Bjorklund (Ed.), *Children's strategies: Contemporary views of cognitive development* (pp. 185-212). Hillsdale, NJ: Lawrence Erlbaum.

Ashcraft, M. H., & Krause, J. A. (2007). Working memory, math performance, and math anxiety. *Psychonomic Bulletin & Review*, 14(2), 243-248.

Barraza, P., Gómez, D. M., Oyarzún, F., & Dartnell, P. (2014). Long-distance neural synchrony correlates with processing strategies to compare fractions. *Neuroscience Letters*, 567, 40-44.

Clarke, D. M., & Roche, A. (2009). Students' fraction comparison strategies as a window into robust understanding and possible pointers for instruction. *Educational Studies in Mathematics*, 72(1), 127-138.

DeWolf, M., & Vosniadou, S. (2015). The representation of fraction magnitudes and the whole number bias reconsidered. *Learning and Instruction*, 37, 39-49.

Gómez, D. M., & Dartnell, P. (2015). Is there a natural number bias when comparing fractions without common components? A meta-analysis. In K. Beswick, T. Muir, & J. Wells (Eds.), *Proceedings of the 39th Psychology of Mathematics Education conference* (Vol. 3, pp. 1-8). Hobart, Australia: PME.

Gómez, D. M., Jiménez, A., Bobadilla, R., Reyes, C., & Dartnell, P. (2014). Exploring fraction comparison in school children. In S. Oesterle, P. Liljedahl, C. Nicol, & D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36* (Vol. 3, pp. 185-192). Vancouver, Canada: PME.

Ischebeck, A., Schocke, M., & Delazer, M. (2009). The processing and representation of fractions within the brain: An fMRI investigation. *NeuroImage*, 47, 403-413.

Kieren, T. E. (1976). On the mathematical, cognitive, and instructional foundations of rational numbers. In R. A. Lesh & D. A. Bradbard (Eds.), *Number and measurement: Papers from a research workshop* (pp. 101-144). Columbus, OH: ERIC/SMEAC.

Ni, Y., & Zhou, Y.-D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist*, 40(1), 27-52.

Obersteiner, A., Van Dooren, W., Van Hoof, J., & Verschaffel, L. (2013). The natural number bias and magnitude representation in fraction comparison by expert mathematicians. *Learning and Instruction*, 28, 64-72.

Reys, R. E., Rybolt, J. F., Bestgen, B. J., & Wyatt, J. W. (1982). Processes used by good computational estimators. *Journal for Research in Mathematics Education*, 13(3), 183-201.

Siegler, R. S., & Stern, E. (1998). Conscious and unconscious strategy discoveries: A microgenetic analysis. *Journal of Experimental Psychology: General*, 127(4), 377-397.

Stafylidou, S., & Vosniadou, S. (2004). The development of students' understanding of the numerical value of fractions. *Learning and Instruction*, 14, 503-518.

Vamvakoussi, X., Van Dooren, W., & Verschaffel, L. (2012). Naturally biased? In search for reaction time evidence for a natural number bias in adults. *The Journal of Mathematical Behavior*, 31, 344-355.

Van Dooren, W., Lehtinen, E., & Verschaffel, L. (2015). Unraveling the gap between natural and rational numbers. *Learning and Instruction*, 37, 1-4.

Zhang, L., Xin, Z., Li, F., Wang, Q., Ding, C., & Li, H. (2012). An ERP study on the processing of common fractions. *Experimental Brain Research*, 217, 25-34.